Scaling and multiscaling in financial time series

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Outline

1/ A brief overview of financial markets

- Basic definitions and problems related to finance
- Scaling in finance

2/ Empirical properties of financial time series

- Main "stylized facts"
- Scaling properties

3/ Empirical models: From Bachelier to Mandelbrot

- Fat tails: Truncated Levy models
- Heteroskedaticity: Classical econometric models.
- Multifractal Models

4/ The MRW model

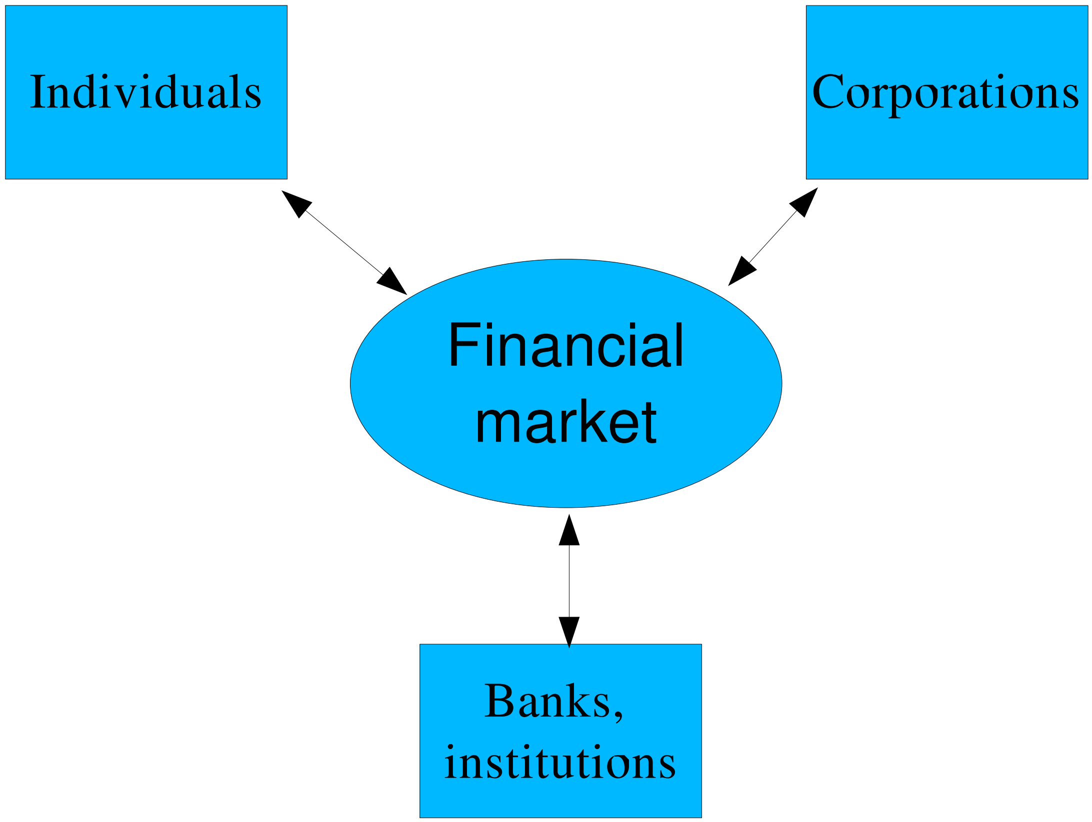
- Definition and scaling properties
- Estimation issues

5/ Applications

- Risk evaluation and forecasting
- Portfolio theory and option pricing

6/ Conclusion and prospects

An overview of financial markets



• Individuals: Speculation, investment

• Corporations, firms: Raise funds (issue shares), investment

• Banks, Financial institutions, Pension funds,...: Hedging, arbitrage

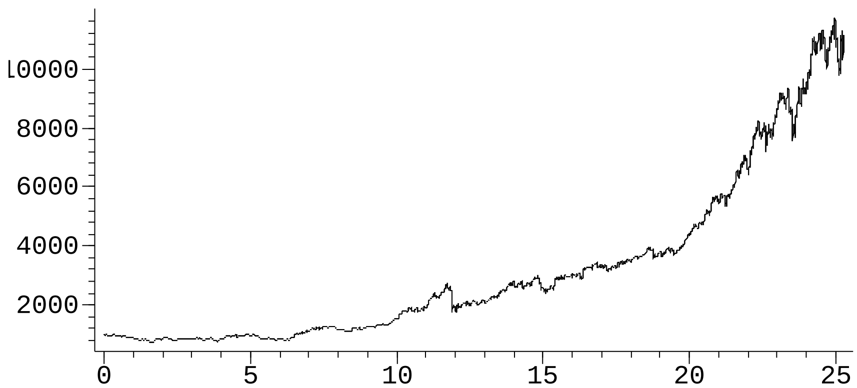
• Markets: Financial securities (Stock, Bonds, options, futures,...), FX rates, Commodities,...

Some definitions: returns

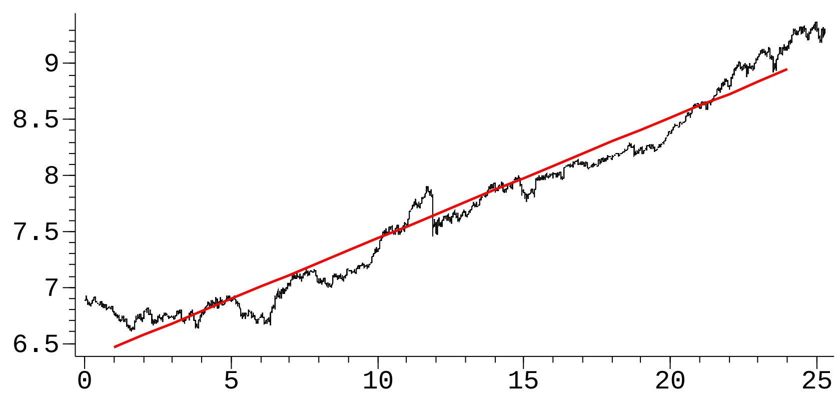
- $P(t)$: market asset *price* at time t
- $X(t) = \ln P(t)$ return process
- Return at time t over a period τ:

$$\begin{aligned} r(t,\tau) &= X(t+\tau) - X(t) = \ln P(t+\tau) - \ln P(t) \\ &\approx \frac{P(t+\tau) - P(t)}{P(t)} = R(t,\tau) \end{aligned}$$

$P(t)$



$X(t)$



Dow-Jones Index

Annual net return: $R \simeq 11\%$

Hold 1 usd over 25 years period $\rightarrow (1.11)^{25} \simeq 13.6$ usd

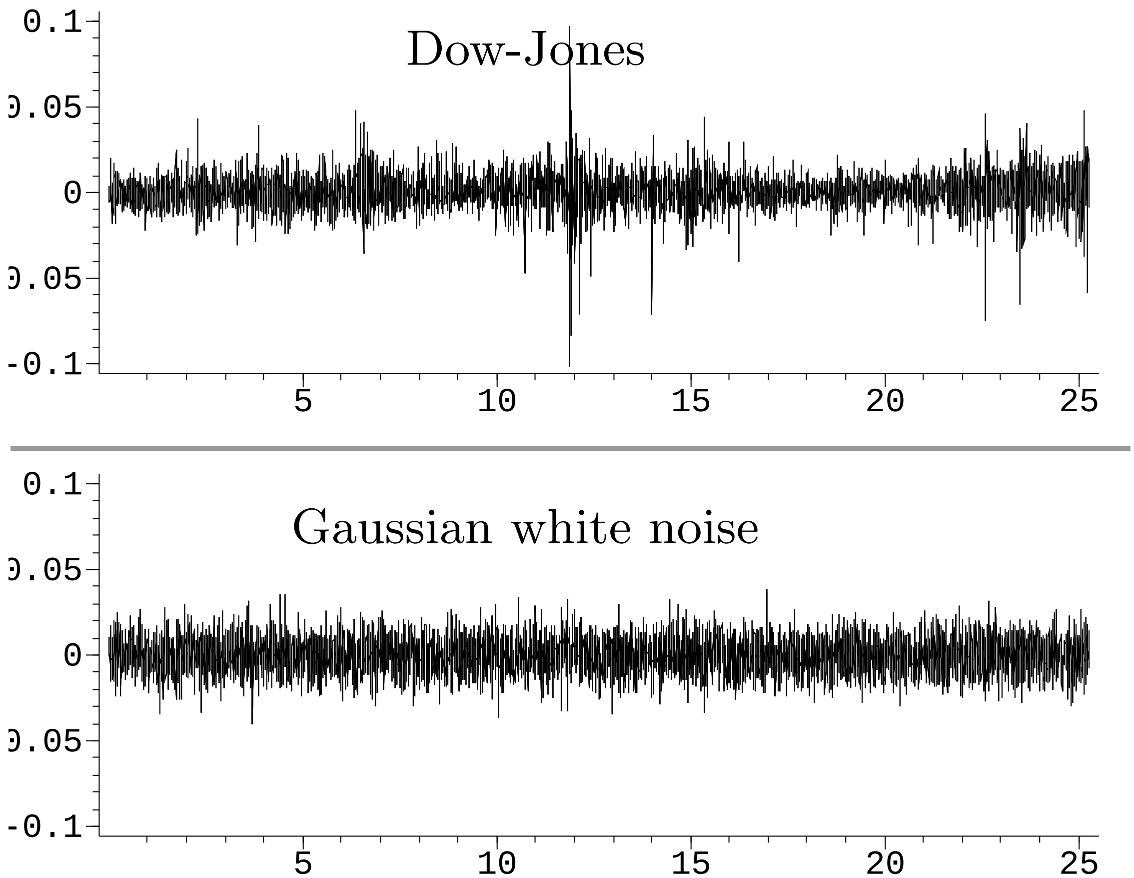
Some definitions: volatility

The *volatility* quantifies the size of return fluctuations.

$$r(t, \tau) = \mu(\tau) + \sigma(\tau)\epsilon(t)$$

where $\epsilon(t)$ is a normalized noise.

It is often identified to the variance σ^2 of return fluctuations.



Daily returns

Problems of quantitative finance

- Rational investment and risk management
  - Price dynamics
  - Risk quantification and control
  - Financial instruments: derivatives
- Micro-economics
  - Market behavior under uncertainty
  - Agent based theory (Utility functions)
- Tools and fields
  - Probability and statistics, time series analysis, stochastic calculus,...
  - Econometrics, applied mathematics, statistical physics, physics of complex systems,...

Scaling in finance

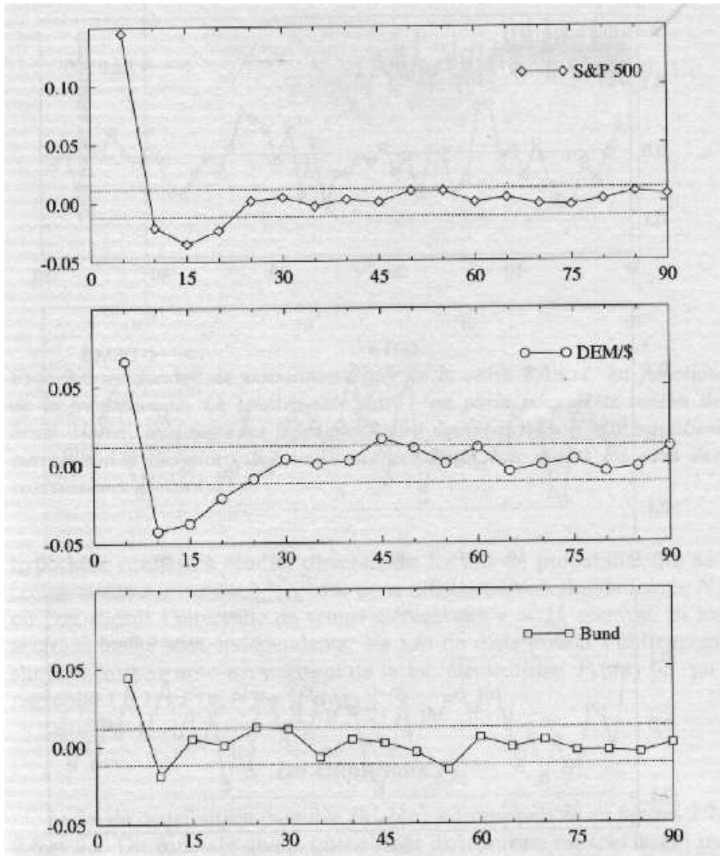
- Supported by empirical observations

- Practical interests.
  - Stability over time scales (by aggregation)
  - The same model is valid over a wide range of scales.
  - Small number of parameters
  - Analytically tractable models

- Theoretical interest: universality and parcimony
  - No preferred scale ratio: scale invariance
  - Continuous time formulation.
  - Universality (small number of pertinent parameters)

Empirical properties of return time series

No return correlation

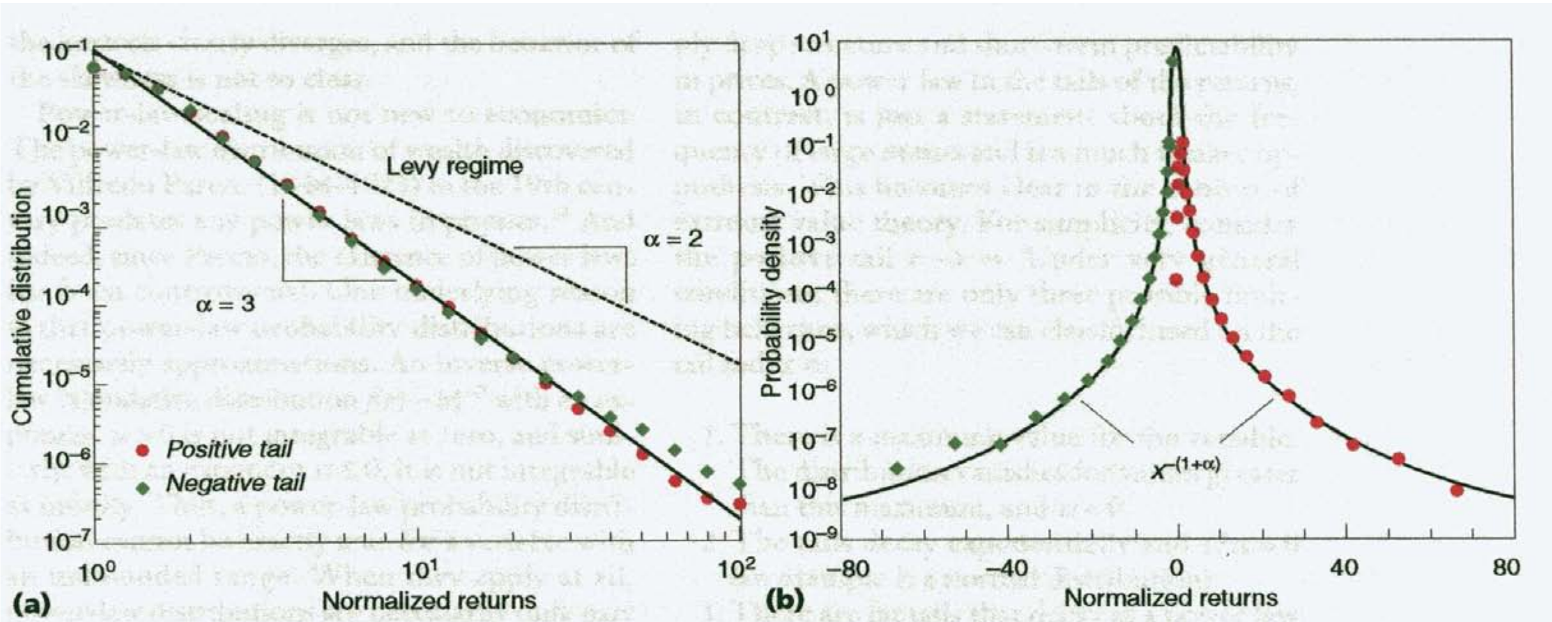


5 min return correlation function associated with 3 assets.
(from *Bouchaud and Potters 1997*)

Market efficiency: Return changes are unforecastable (martingale hypothesis)
⇒ Prices are *linearly* unforecastable

Empirical properties of return time series

- Return pdf have fat tails at small scales

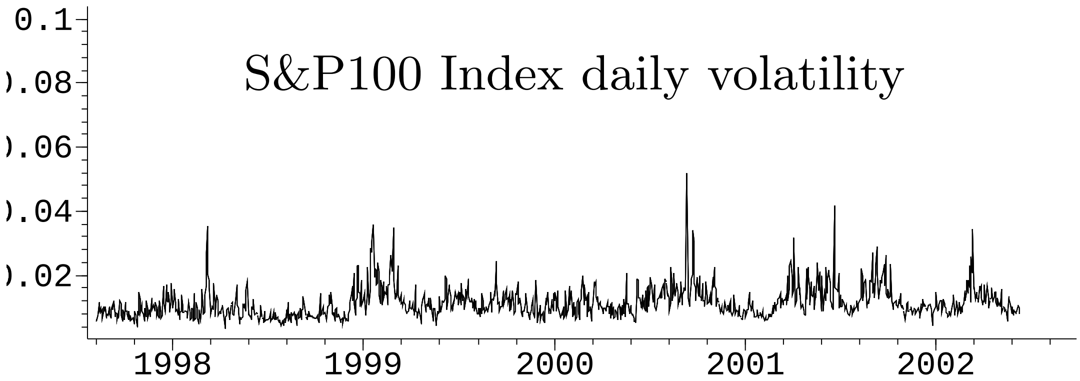


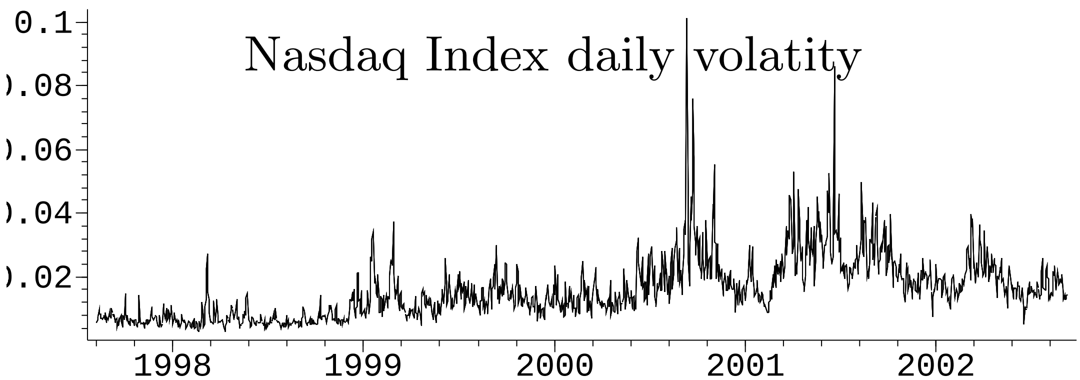
Cumulative distribution and pdf of normalized 5-min returns of 1,000 largest US companies. (from *Farmer 1999*)

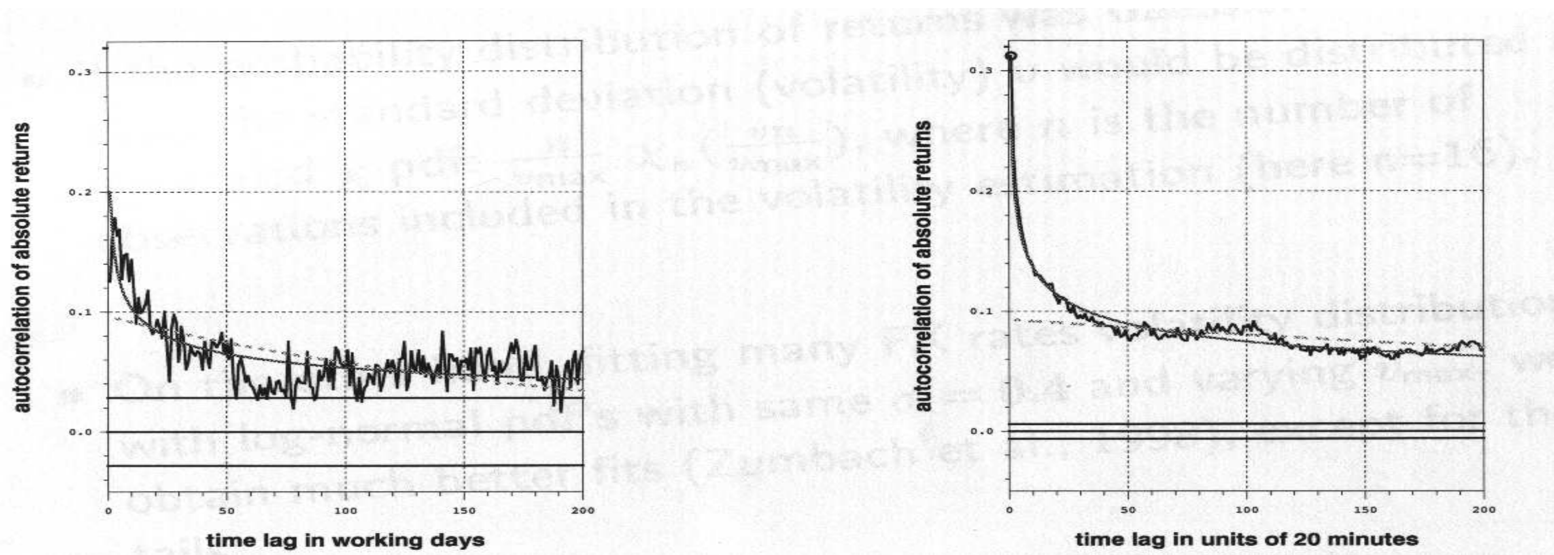
- Quasi-Gaussian at large scales

Empirical properties of return time series

Volatility clustering





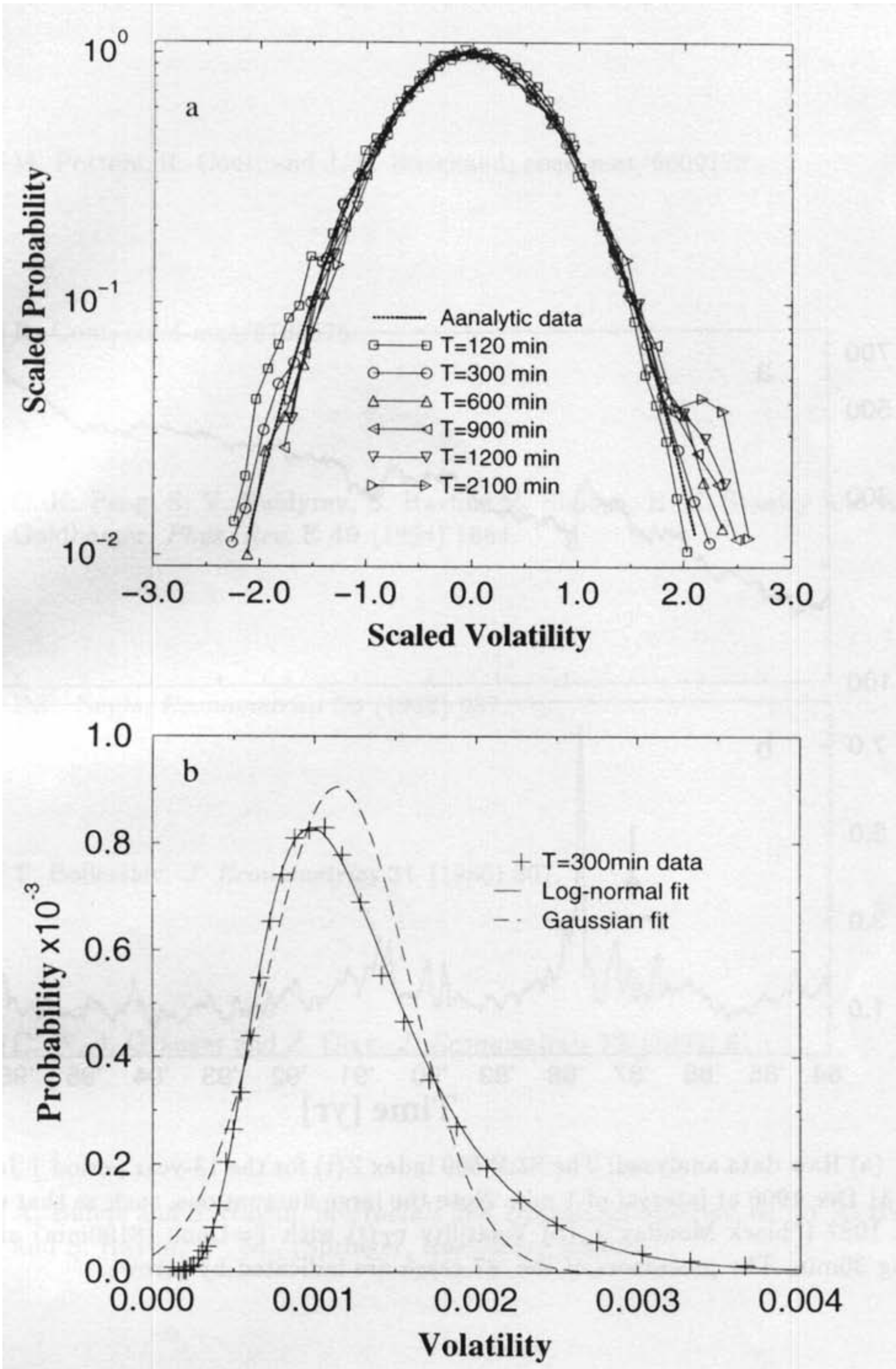


Power-law volatility correlation of USD-DM rate.

(from *Daracogna 2000)*

Empirical properties of return time series

Log-normal volatility



S&P500 Volatility distribution.

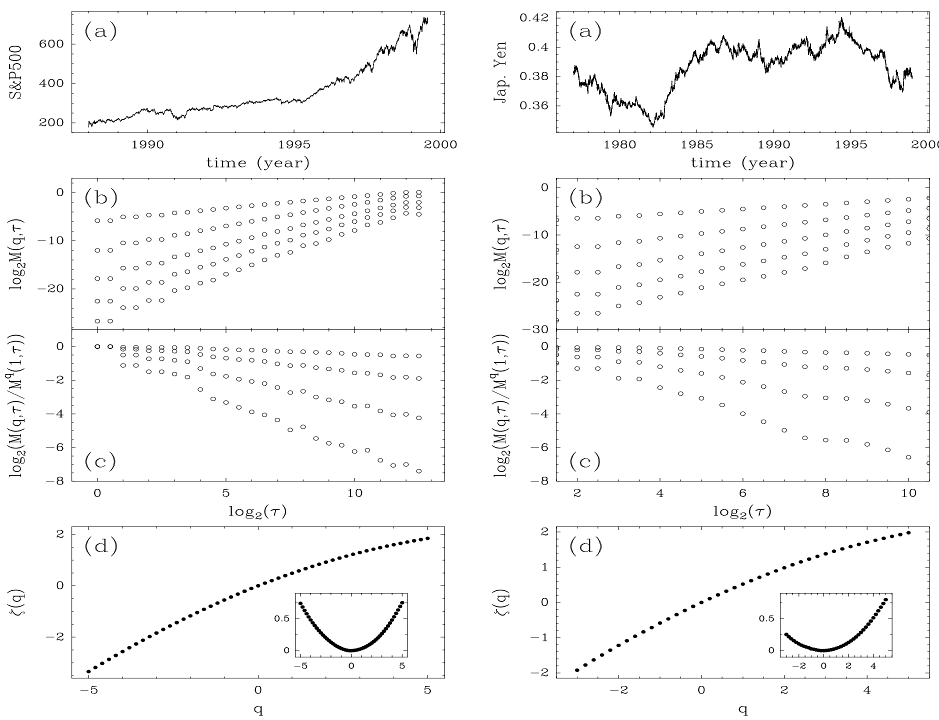
(from *Cizeau et al. 1997)*

Empirical properties of return time series

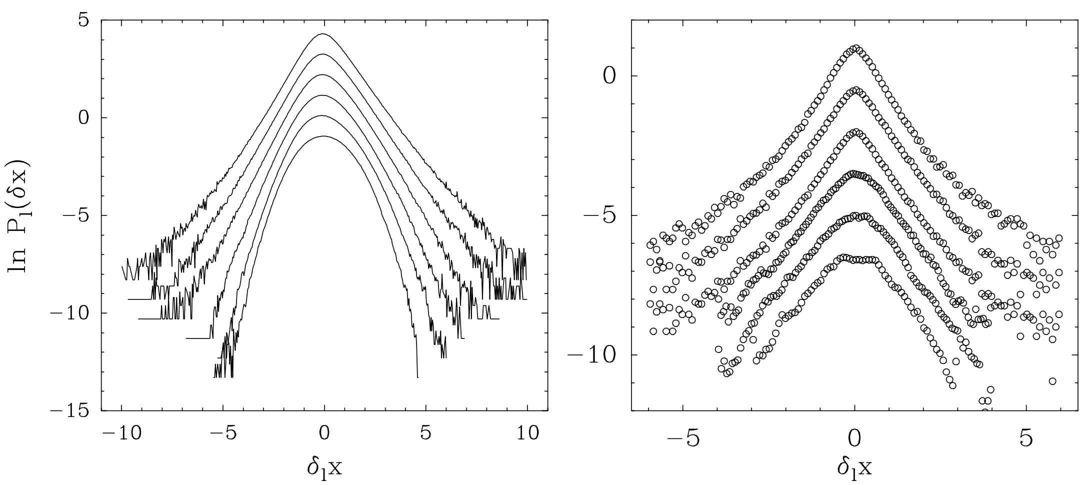
Multi-scaling of returns

- Scaling of return absolute moments

$$M(q,\tau) = \mathbb{E}\left[|r(\tau,t)|^q\right] \sim K_q \tau^{\zeta_q}.$$



- The return pdf varies strongly across scales

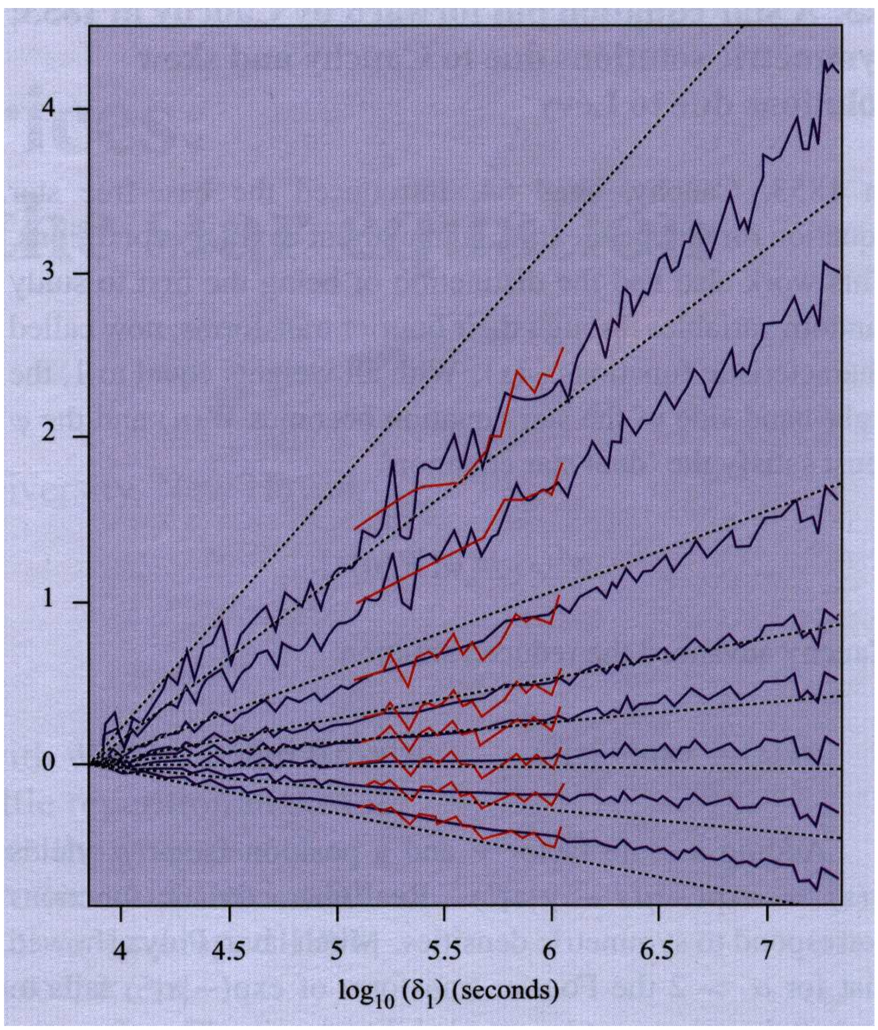


Empirical properties of return time series

Multi-scaling of returns

FX rates: Ghashgaie, Breynmann, Peinke, Talkner (1996), Calvet, Fisher, Mandelbrot (1997), Schmitt, Schertzer, Lejevoy (1998), Vandewalle, Ausloos (1998),...
Stock markets: Brachet, Taflin, Tcheou (1997), Ausloos, Ivanova (2001), Bershaskii (2001),...
Future markets: Arneodo, Muzy, Sornette (1998), Muzy, Delour, Bacry (2000)...

$$M(q, \tau) = \mathbb{E}\left[|r(\tau, t)|^q\right] \sim K_q \tau^{\zeta_q}.$$



Multiscaling of USD-DM FX rate (from Mandelbrot 2002)

Empirical models for return fluctuations

Bachelier model (1900)

The return process $X(t) = \ln P(t)$ is a Brownian motion:

$$dX(t) = \mu \, dt + \sigma \, dW(t)$$

- No correlation
- "Everything" can be computed (stochastic calculus,...)
- Universality

but

- Gaussian law at all scales: no fat tails
- Constant volatility: no volatility aggregation
- Simple scaling properties (self-similarity)

This model is still at the heart of most models used financial engineering.

Empirical models for return fluctuations

(Truncated) Levy models

• α-stable process (Mandelbrot, Fama, 1963)
The return process $X(t)$ satisfies:

$$dX(t) = \mu \, dt + \sigma \, dL_\alpha(t)$$

where $L_\alpha(t)$ is an α-stable Levy process. The returns $r(\tau, t)$ have therefore α-stable laws.

- Fat tails
- Lot of possible computations
- Multi-scaling ("bi-scaling")

but

- The variance is infinite
- Jumps
- No volatility clustering

• Truncated α-stable process (Mantegna & Stanley, 1995):
The stable law is exponentially truncated in the tail.

Empirical models for return fluctuations

(G)ARCH models (*Engle 1982, Bollersev 1986*)

The return at scale τ, $r(n\tau,\tau)$, is conditionally Gaussian:

$$r(n\tau,\tau) \equiv r(n) = \sigma(n)\epsilon(n)$$

where $\epsilon(n)$ is a Gaussian white noise and the volatility $\sigma^2(n)$ is a regression from past squared returns and volatilities:

$$\sigma^2(n) = \alpha_0 + \sum_{i=1}^{p} \alpha_i r^2(n-i) + \sum_{j=1}^{q} \beta_j \sigma^2(n-i)$$

- Volatility clustering
- Easy to estimate (M.L.)
- Leptokurticity (heavy tail)

but

- Volatility correlations decrease rapidly
- No (multi-) scaling property
- Discrete time model (parameters change across scales)

GARCH(1,1) ($p = q = 1$) is a very popular model for volatility forecasting.

Empirical models for return fluctuations

Stochastic volatility models

(*Taylor 1985, Hull & White 1987*)

The return at scale τ, $r(n\tau, \tau)$, is conditionally Gaussian:

$$r(n\tau, \tau) \equiv r(n) = \sigma(n)\epsilon(n)$$

where $\epsilon(n)$ is a Gaussian white noise and the volatility $\sigma^2(n)$ is itself a random process.

Usually $\omega(n) = \ln \sigma^2(n)$ is chosen to be AR(1):

$$\omega(n) = \phi\, \omega(n-1) + \nu(n)$$

where $\nu(n)$ is a Gaussian white noise independent of $\epsilon(n)$

- Volatility clustering
- Easy to estimate (no exact M.L.)
- Leptokurticity

but

- Volatility correlations decrease rapidly
- Discrete time model (parameters change across scales)
- No multiscaling

Empirical models for return fluctuations

Multifractal models

$$\boxed{\mathbb{E}\left[|X(t)|^q\right] = K_q t^{\zeta_q}}$$

• MMAR model (*Calvet, Fischer, Mandelbrot, 1999*)
The return $X(t) = \ln P(t)$ is a Brownian motion compound with a multifractal "time" $M(t)$:

$$\begin{aligned} X(t) &= B\left[M(t)\right] \\ M(t) &\equiv \text{Multiplicative cascade} \end{aligned}$$

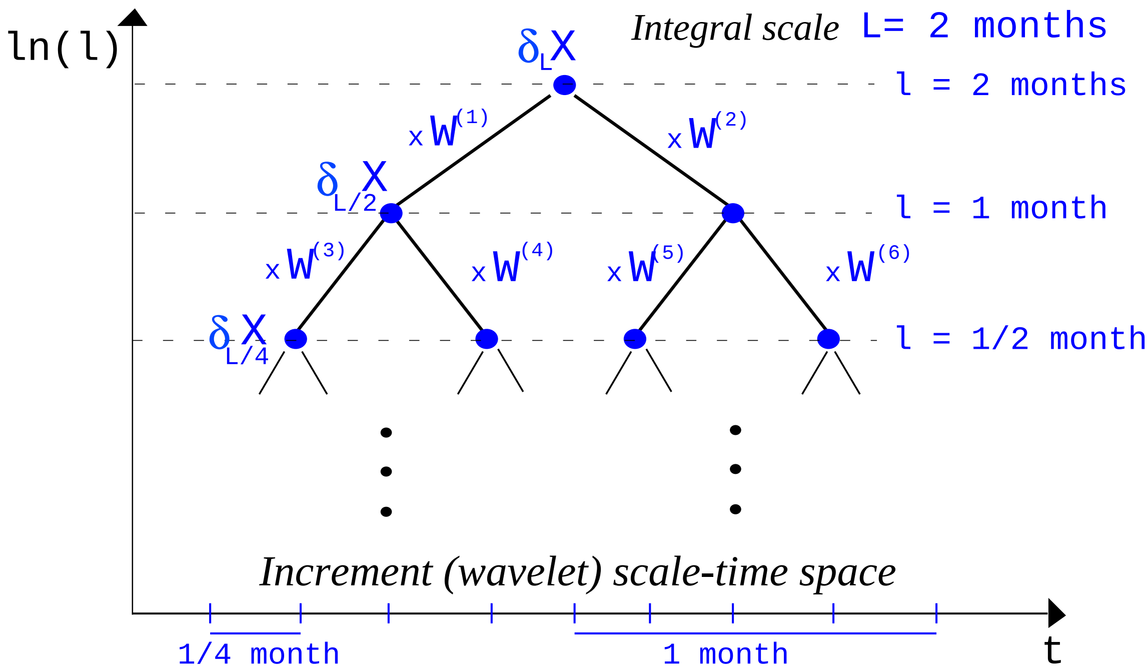
• MRW model (*Bacry, Delour, Muzy, 2000*)
The return $X(t)$ is obtained as the continous limit of a stochastic variance model:

$$\begin{aligned} X_l(t) &= \sum_{k=1}^{t/l} \sigma_l(k)\epsilon_l(k) \\ M(t) &= \int_0^t \sigma^2(u)\, du \end{aligned}$$

The MRW model

From Discrete cascades to stochatic variance models

$$\boxed{\{r_{\lambda l}(\lambda t)\}_t = \lambda^H \{r_l(t)\}_t = W_\lambda \{r_l(t)\}_t}$$



- Volatility Magnitude : $\omega_l(t) = \frac{1}{2} \ln |r_l(t)|^2$
- Magnitude diffusion from coarse to fine scales:

$$\omega_{2^{-n-1}} = \omega_{2^{-n}} + \epsilon_{2^{-n-1}}$$

with $\epsilon = \ln(W)$ and $\lambda^2 = \mathrm{Var}(\epsilon)$

- Ultrametric (tree) structure

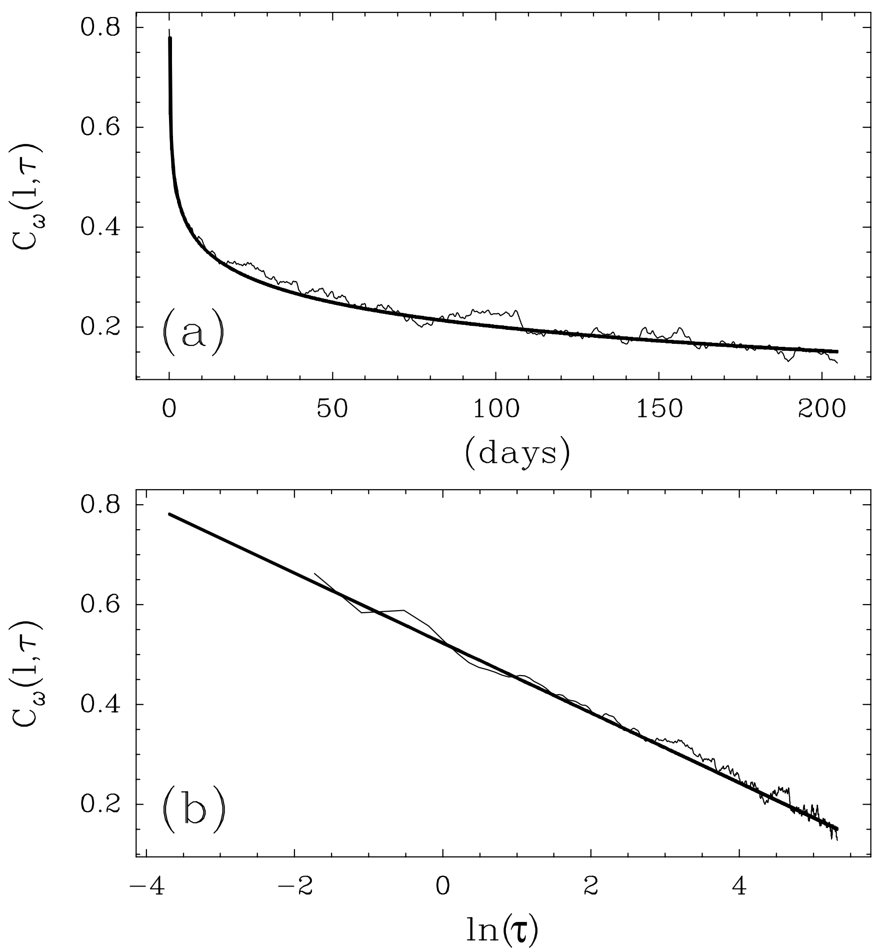
Arneodo, Muzy, Sornette, 98

Arneodo, Bacry, Muzy, Manneville, 98

$$\boxed{\mathbb{C}\mathrm{ov}\left(\omega_l(t), \omega_l(t+\tau)\right) \simeq -\lambda^2 \ln(\tau/T), \quad l << \tau < T}$$

Magnitude correlation of the S&P 500 futures

Arneodo, Muzy, Sornette, 98



The MRW model

(Bacry, Delour, Muzy 2000)

$$
\begin{aligned}
X(t) &= \lim_{l \to 0} X_l(t) \\
X_l(t) &= X_l(t-l) + \sigma_l(t)\epsilon_l(t)
\end{aligned}
$$

$\epsilon_l(t)$: Gaussian white noise

$\sigma_l = e^{\omega_l}(t)$: Stochastic volatility

$\omega_l(t)$: Gaussian (inf. div.) log-correlated magnitude:

$$\mathbb{C}\mathrm{ov}\,(\omega_l(t), \omega_l(t+\tau)) \simeq -\lambda^2 \ln(\tau/T), \quad l < \tau \leq T$$

$$\Updownarrow$$

- Stationary (uncorrelated) increments
- Multifractal process
- Continuous scale invariance properties
- Fat tails
- Quasi-lognormal volatilities for $\lambda^2 << 1$
- Volatility clustering

Only 3 parameters: noise variance σ^2, intermittency parameter λ^2 and volatility correlation time T .

Multifractal scaling properties of MRW

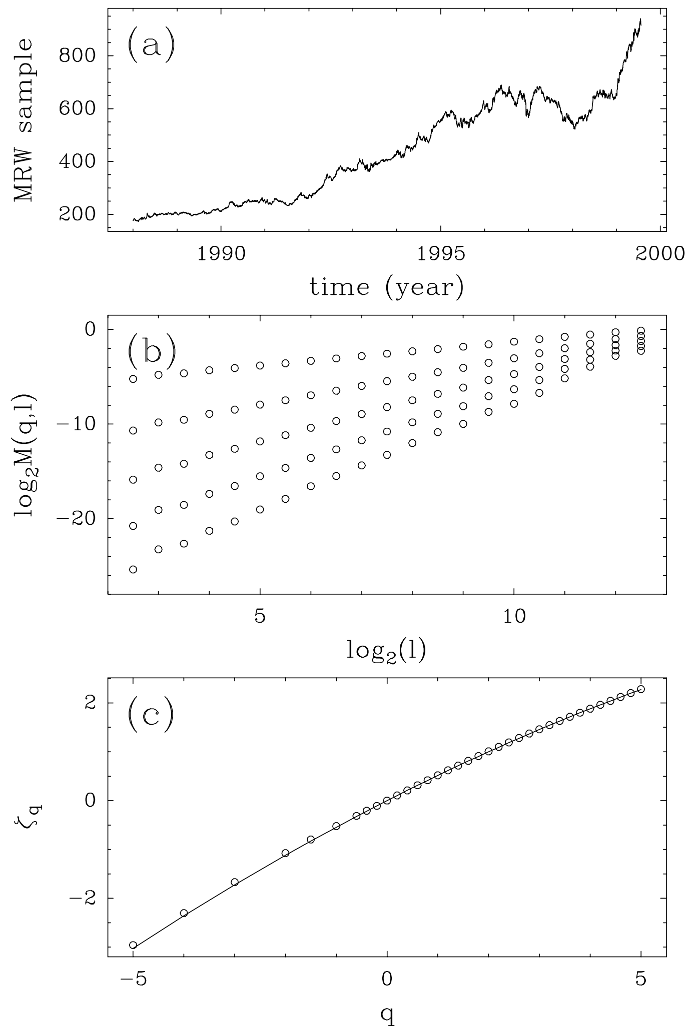
(Bacry, Delour, Muzy, 2000)

$$r(\lambda\tau, \lambda t) =^{\mathcal{L}} e^{\Omega_\lambda} r(\tau, t)$$

$$\mathbb{E}\left[|r(\tau,t)|^q\right] = K_q \left(\frac{\tau}{T}\right)^{\zeta_q}$$

$$\zeta_q = \frac{q}{2} - \lambda^2 q(\frac{q}{2} - 1)$$

$$\zeta_q < 1 \text{ ``}\Leftrightarrow\text{''} \; K_q = +\infty$$



Analytical expression for the factors K_{2n}:

$$K_{2n} = T^n \sigma^{2n} (2n-1)!! \prod_{k=0}^{n-1} \frac{\Gamma(1-2\lambda^2 k)^2 \Gamma(1-2\lambda^2(k+1))}{\Gamma(2-2\lambda^2(q/2+k-1))\Gamma(1-2\lambda^2)}$$

Fixed scale MRW returns

(Bacry, Khozemyak, Muzy, 2004)

- Fixed scale return ($\Delta < T$):

$$r_\Delta(k) \equiv r(\Delta, k\Delta) = X\left((k+1)\Delta\right) - X(k\Delta)$$

- Stochastic volatility

$$r_{\Delta k}(k) \stackrel{\mathcal{L}aw}{=} \varepsilon(k) e^{\Omega_\Delta[k]}$$

where $\varepsilon[k]$: gaussian white noise.

- In the small intermittency limit $\lambda^2 << 1$:

$$\Omega_\Delta(k) \approx \lambda \Gamma_\Delta(k)$$

where $\Gamma_\Delta(k)$ is a known Gaussian process (*"renormalized magnitude"*)

- Moreover, if $R_\Delta(k) = \varepsilon(k) e^{\lambda \Gamma_\Delta(k)}$

$$\mathbb{E}\left[(\ln|r_\Delta(k_1)|)^{p_1} \ldots\right] = \mathbb{E}\left[(\ln|R_\Delta(k_1)|)^{p_1} \ldots\right]\left(1 + o(\lambda^{2-\epsilon})\right)$$

- If $\mathbb{E}\left[|r_\Delta(k_1)|^{q_1} \ldots\right] < +\infty$,

$$\mathbb{E}\left[|r_\Delta(k_1)|^{p1} \ldots\right] = \mathbb{E}\left[|R_\Delta(k_1)|^{p1} \ldots\right]\left(1 + o(\lambda^{2-\epsilon})\right)$$

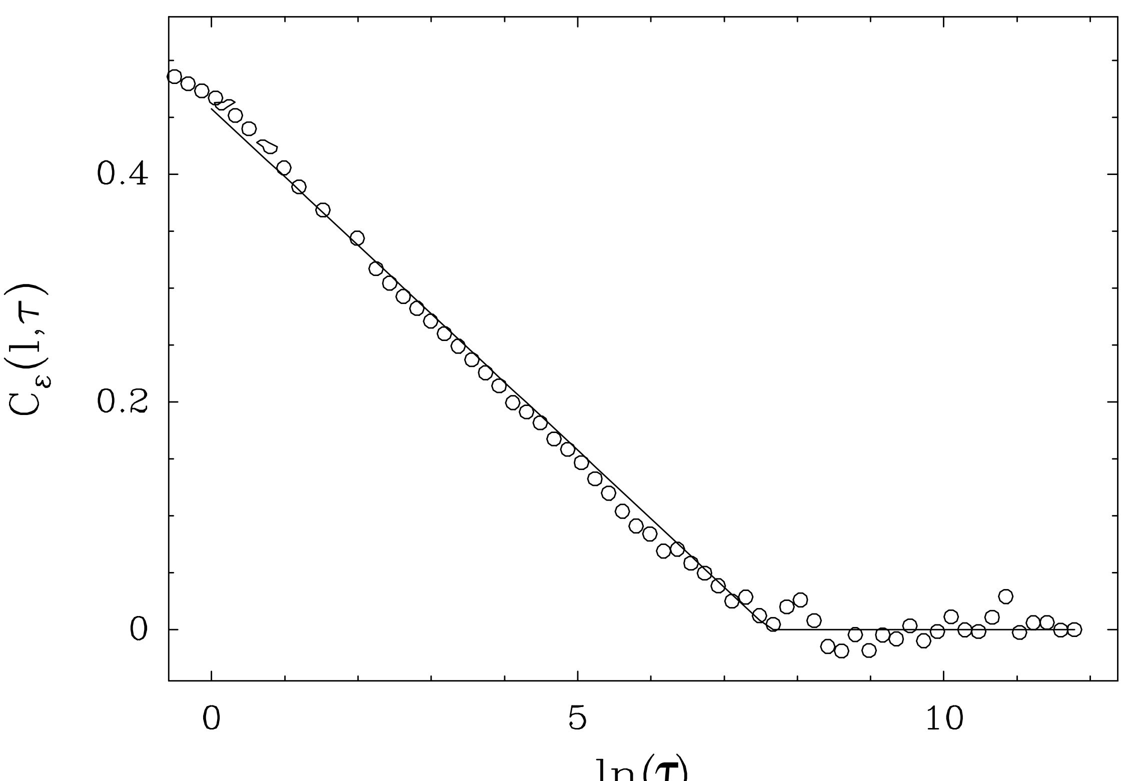
Magnitude correlation

$$\ln|r_\Delta(k)| \overset{\mathcal{M}}{\simeq} \ln|\varepsilon_\Delta(k)| + \lambda\Gamma_\Delta(k)$$

$$g(n) = n^2 \ln(n)$$

$$\mathbb{C}\mathrm{ov}\,(\Gamma_\Delta(k), \Gamma_\Delta(k+n)) = \ln(\frac{Te^{3/2}}{\Delta}) + g(n) - \frac{1}{2}(g(n+1) + g(n-1))$$

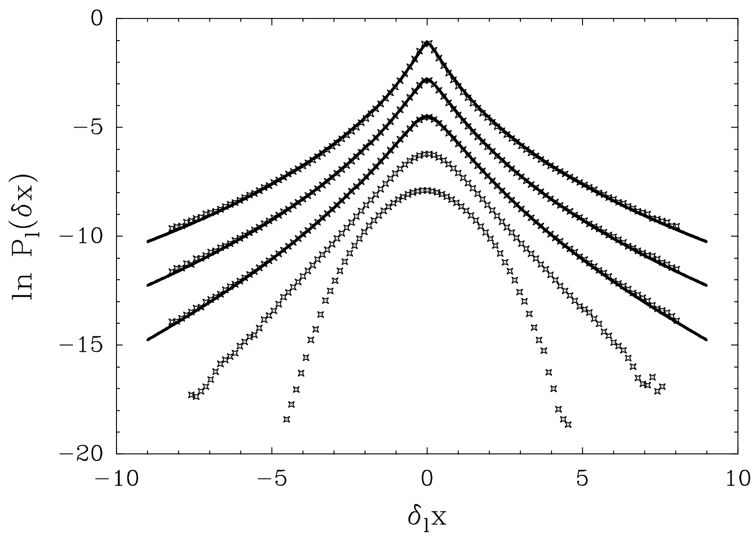
$$\sim \ln(\frac{T}{n\Delta}) \text{ when } n \gg 1$$



Continuous deformation of increment pdf's across scales

(Bacry, Delour, Muzy, 2000)

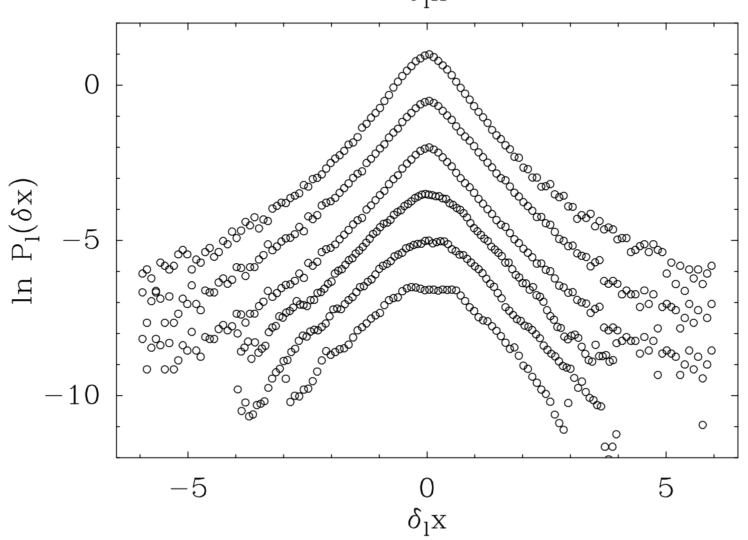
$$r_\Delta(k) \overset{\mathcal{L}aw}{\simeq} \varepsilon_\Delta(k) e^{\lambda \Gamma_\Delta(k)}$$



MRW

∘∘∘∘ Numerical estimation

—— Castaing prediction



S&P Futures

∘∘∘∘ Numerical estimation

Multifractal estimation issues

• Small number of studies devoted to statistical estimation of random cascades.

• Finance:

- Heuristical, Monte-Carlo estimates from "log-log"
- GMM method from binomial cascade moments (*Lux 2002*)

• MRW:

- "Integral time" T

$\longrightarrow$ Decorrelation time

- Variance σ^2

$\longrightarrow$ return variance

- Intermittency coefficient λ^2

$\longrightarrow$ log-volatility correlation $\ln|r_\Delta(t)|$:

$$\mathbb{C}\text{ov}\left(\ln r_\Delta(t), \ln r_\Delta(t+\tau)\right) \simeq -\lambda^2 \ln(\tau/T), \quad \Delta < \tau < T$$

$\longrightarrow$ Absolute moments:

$$\mathbb{E}\left[|r_\Delta|^{2q}\right] = K_{2q} T^q \sigma^{2q} (2q-1)!! \left(\frac{l}{T}\right)^{\zeta_{2q}}$$

$\Rightarrow$ G.M.M. method

GMM estimation

Principle (*Hansen 1982*)

$\{r(k)\}$: data
$\vec{\theta}$: vector of p parameters to be estimated
$\mu_j(\{r(k)\},\vec{\theta})$: "moments" satisfying the condition:

$$\mathbb{E}\left[\mu_j(\{r(k)\},\vec{\theta})\right] = 0 \text{ for } 1 \leq j \leq m,\ m > p$$

Let $\bar{\mu}_j(\vec{\theta})$ a sample estimator of $\mathbb{E}\left[\mu_j(\{r(k)\},\vec{\theta})\right]$. Then the GMM estimator of $\vec{\theta}$, $\hat{\theta}$ is obtained as

$$\hat{\theta} = \arg\min_{\vec{\theta}}\left[\bar{\mu}_i(\vec{\theta}) W_{ij}^{-1} \bar{\mu}_j(\vec{\theta})\right]$$

where W is a positive definite weighting matrix $m \times m$ matrix. When $W_{ij} = \mathbb{E}\left[\bar{\mu}_i \bar{\mu}_j\right]$ (or a consistent estimate of it), the GMM estimator is consistent and asymptotically efficient. Moreover,

$$\sqrt{N}(\hat{\theta} - \vec{\theta}) \underset{\mathcal{L}aw}{\rightarrow} \mathcal{N}(\vec{0}, \Sigma(\vec{\theta}))$$

with

$$\Sigma(\vec{\theta})_{ij} = \frac{\partial \bar{\mu}_i}{\partial \theta_k} W_{kl}^{-1} \frac{\partial \bar{\mu}_k}{\partial \theta_j}$$

$\rightarrow$ Confidence intervals, tests,...

GMM estimation of MRW

(*Bacry, Muzy 2004*)

$\{r_\Delta(k)\}$: return data
$\vec{\theta}$: vector (σ^2, λ^2, T)
$\mu_j(\{r_\Delta(k)\}, \vec{\theta})$: "moments"

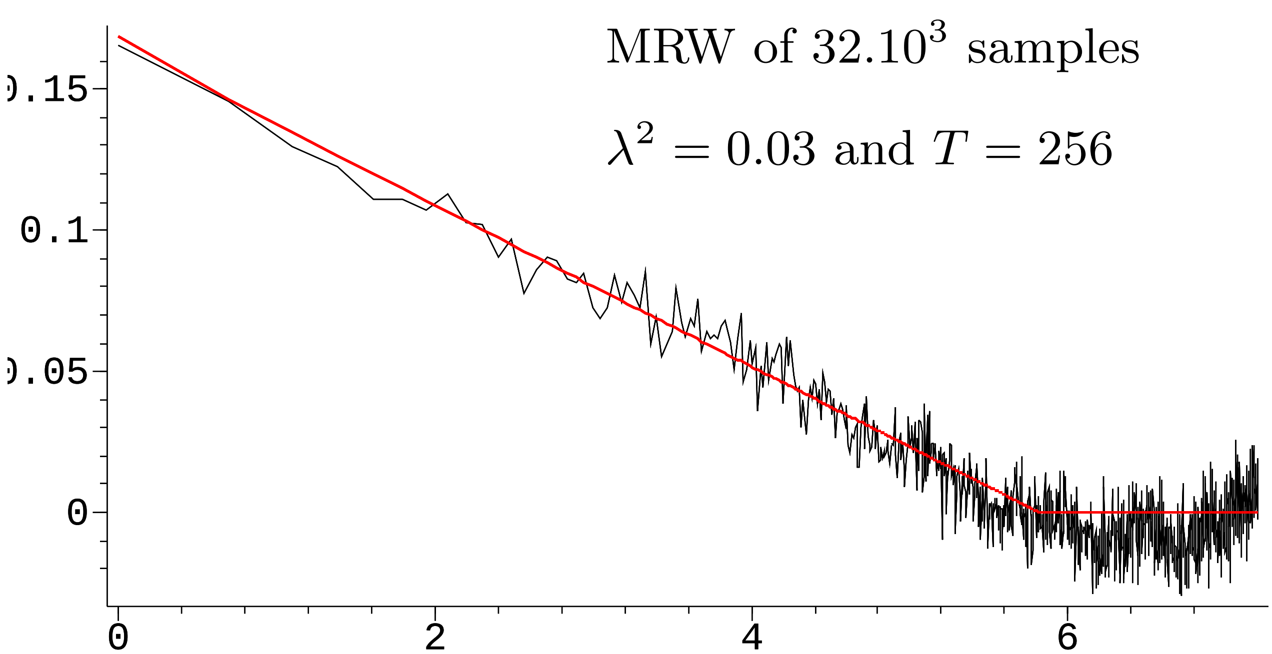
$$\mu_j = r_\Delta(k)^{2n} - M(\sigma^2, \lambda^2, T, n)$$
$$\mu_j = \ln|r_\Delta(k)| \ln|r_\Delta(k+n)| - C(\sigma^2, \lambda^2, T, n)$$

satisfying the condition:

$$\mathbb{E}\left[\mu_j(\{r(k)\}, \vec{\theta})\right] = 0 \text{ for } 1 \leq j \leq m.$$

One then use an iterative procedure:

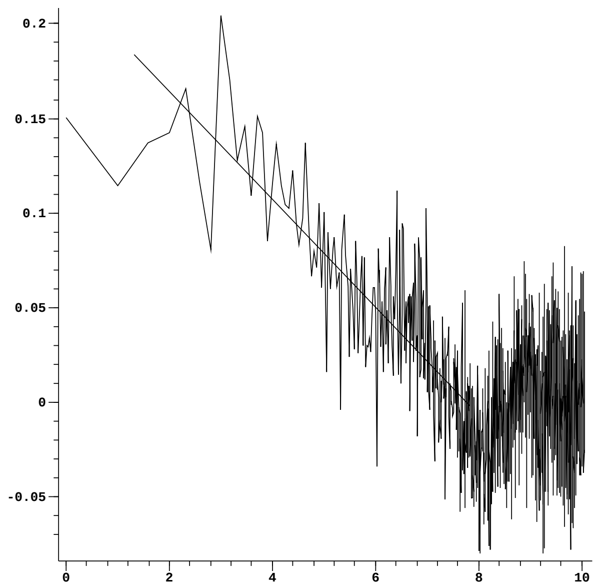
$$W_0 = \mathcal{I} \to (\sigma_0^2, \lambda_0^2, T_0) \to W_1 \to (\sigma_1^2, \lambda_1^2, T_1) \to \dots$$



5% confidence intervals $\lambda^2 \in [0.025, 0.032]$ and $T \in [225, 503]$.

Parameter estimation for daily data

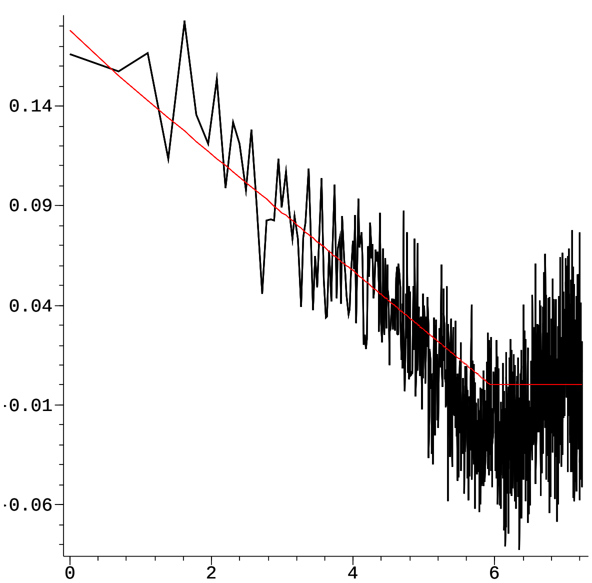
(Bacry, Muzy, 2002)



CAC 40 Index daily data

1/1/1973 to 31/12/97 (6239 points)

$\lambda^2 = 0.03 \pm 0.01$, $T \in [0.5, 2]$ years



MRW sample

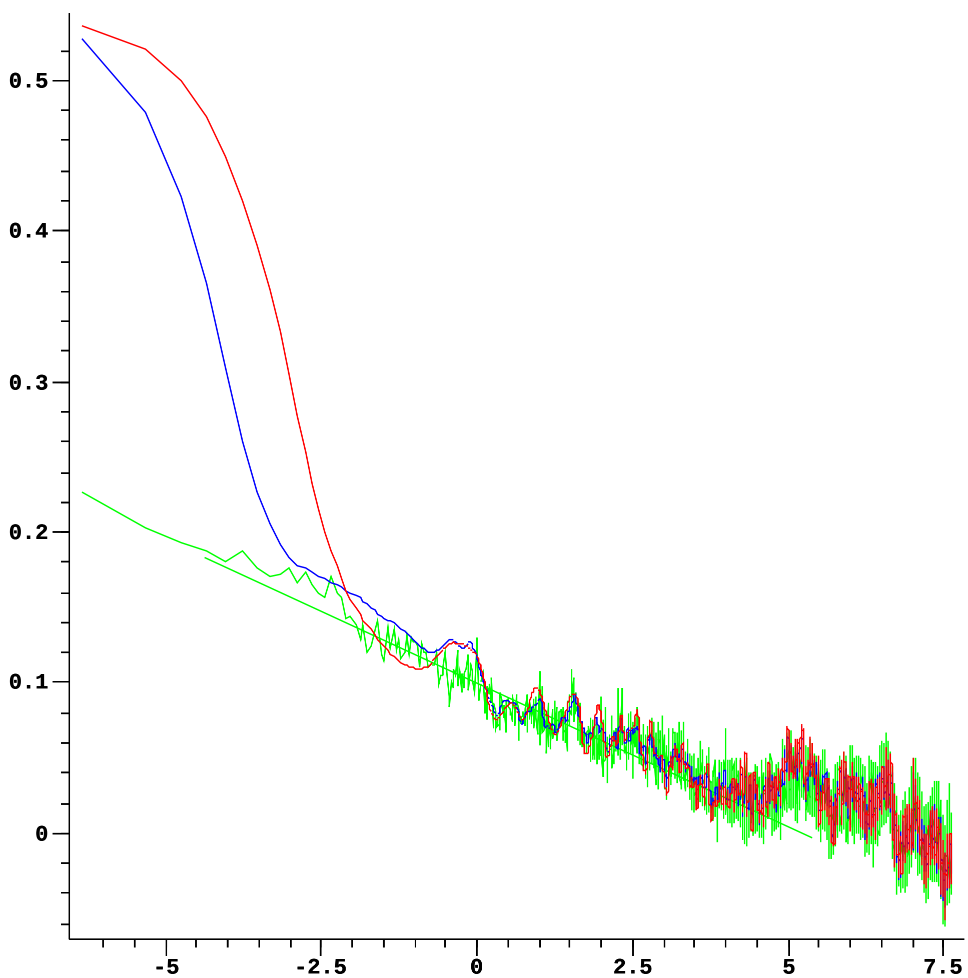
5000 points

$\lambda^2 = 0.03$, $T = 250$

Parameter estimation intraday data

(Bacry, Muzy, 2002)

S&P 500 intraday data (5mn ticks, 1996-1998)



: 5mn returns

: 30mn returns

: 1h returns

Parameter values for financial returns

(Bacry, Muzy, 2002)

Series	Size	λ^2	T
S&P500 index	7.10^4	0.03	6 months
Future S&P500	7.10^4	0.025	2 years
FTSE100 index	7.10^4	0.028	1.2 year
Future FTSE100	7.10^4	0.029	1 year
Future JY/USD	7.10^4	0.02	6 months
Nikkei 225	7.10^4	0.030	1.5 year
Future Nikkei	7.10^4	0.02	6 months
Japanese Yen	7.10^4	0.022	1.0 year
French index	6.10^3	0.029	1 year
Italian index	6.10^3	0.029	2 years
Canadian index	6.10^3	0.032	1.5 years
German index	6.10^3	0.027	2 years
UK index	6.10^3	0.023	2 years
hong-kong index	6.10^3	0.037	3 years

Generic values: $\lambda^2 = 0.03$, $T = 1$ year.

Applications

Risk evaluation and forecasting

- Volatility estimation and forecasting:
  - Model evaluation
  - Risk estimates
  - Fund manager comparison
  - Active trading
  - Option markets

- Classical econometric models
  - GARCH: Regression from past square returns and volat.
  - J.P. Morgan RiskMetrics: "Optimal" exponential smoothing of past square returns
  - Stochastic volatility: Wiener or Kalman filtering

MRW volatility prediction

(Bacry, Muzy, 2002)

$$\boxed{r_\Delta(k) = \varepsilon_\Delta[k] e^{\Omega_\Delta[k]}}$$

- "Generic" estimates : $\lambda^2 = 0.03$, $T = 1$ year.

- Volatility prediction at scale $\Delta_1 = l\Delta$ $(l \geq 1)$ (Wiener filtering)

  – Method *MRWlin*: Prediction of $\sigma_{\Delta_1}[n] = e^{\Omega_{\Delta_1}[n]}$

  $\hat{\sigma}_{\Delta_1}[n]^2 = [h_1 * r_\Delta^2](n)$ $(h_1$ causal$)$

  – Method *MRWlog* : Prediction of $\Omega_{\Delta_1}[n]$

  $\hat{\omega}_{\Delta_1}[n] = [h_2 * \ln(|r_\Delta|)](n)$ $(h_2$ causal$)$

  $\longrightarrow$ MLE of $e^{\omega_{l_1}[n]}$

- Testing the prediction:

  - MSE (L2) Error :

$$e_{MSE}^2 = \mathbb{E}\left[(\hat{\sigma}_{\Delta_1}[n]^2 - \textstyle\sum_{i=1}^{l} |r_\Delta(i)|^2)^2\right]$$

  - MAE (L1) Error :

$$e_{MAE} = \mathbb{E}\left[|\hat{\sigma}_{\Delta_1}[n]^2 - \textstyle\sum_{i=1}^{l} |r_\Delta(i)|^2|\right]$$

Volatility prediction

(Bacry, Muzy, 2002)

Comparisons on 10 daily Index series

($h, s \subset$ [1 day, 10 days, 1 month, 6 months])

- MSE (L2) Error : Number of "hits"

  MRWlog = 66 , GARCH = 28 , RM = 6, Hist = 0

  MRWlin = 81 , GARCH = 19 , RM = 0, Hist = 0

  MRWlog = 57 , MRWlin = 43 , RM = 0, Hist = 0

- MAE (L1) Error : Number of "hits"

  MRWlog = 80 , GARCH = 20 , RM = 0, Hist = 0

  MRWlin = 88 , GARCH = 12 , RM = 0, Hist = 0

  MRWlog = 72 , MRWlin = 28 , RM = 0, Hist = 0

$\Longrightarrow$ MAE : MRWlog (or MRWlin)

$\Longrightarrow$ MSE : MRWlin

Using intraday data for Volatility prediction

SP100 index : 04/08/97 – 17/12/01 (intraday 5mn)

h=1 day, s = ...	MRWlog (daily)	MRWlog (intra)
RMSE 1 day	+2.33	**-4.30**
10 days	+10.65	**-15.79**
1 month	+15.55	**-16.93**
MAE 1 day	**-13.85**	-10.88
10 days	-4.55	**-24.88**
1 month	-0.80	**-28.00**

Value at risk forecasting

- Definition of the Value at Risk $V(p)$ at level p:

$$\mathbb{P}\left(r_\Delta \leq -V(p)\right) = p$$

$\rightarrow$ Intuitive interpretation: Most probable amplitude of the worst loss at scale Δ over an horizon p^{-1} periods.

Country	Gaussian VaR	Observed
France	0.59	1.24
Japan	0.65	1.08
USA	1.85	2.26
GB	1.59	2.08

Gaussian most probable worst day versus observed worst day during the year 1994 for some international bond indices.

- Usage
  - Objective tool, easy interpretation
  - Widely used in performance evaluation
  - Optimization of risk allocation in a non Gaussian world
- Computation: Historical, Monte-Carlo, Analytical,...

$\rightarrow$ Conditional Gaussian (Garch, RiskMetrics,...)

Value at risk forecasting using MRW

(Bacry, Khozemyak, Muzy, 2004)

- Normal law: $n(x) \equiv \mathcal{N}(0,1)$ and $N(x) = \int_{-\infty}^{x} n(u)du$.
- Castaing formula for probability distribution:

$$r_\Delta(k) \stackrel{\mathcal{L}aw}{=} \varepsilon_\Delta(k) e^{\Omega_\Delta(k)}$$

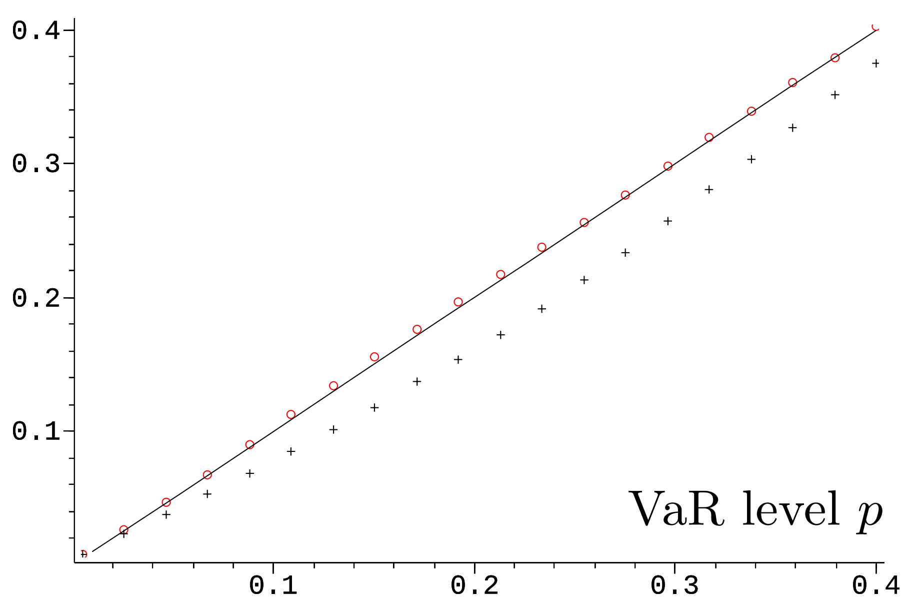
$$\mathbb{P}\left(r \leq x\right) = \int_{-\infty}^{+\infty} g_\lambda(u) N(e^{-u}x)\, du$$

- "log-normal" volatility when $\lambda^2 \to 0$ (Hedgeworth expansion)

$$\frac{\Omega_\Delta(k)}{\lambda} \stackrel{\mathcal{L}aw}{\to} \mathcal{N}\left(0, \ln \frac{Te^{3/2}}{\Delta}\right)$$

$$\lambda g_\lambda(\lambda u) = n(u) + n'(u)\left(\lambda p_1(u) + \lambda^2 p_2(u) + \ldots\right)$$

VaR prediction backtesting for BP-USD rate



○ MRW, + Garch(1,1), − exact.

Volatility dynamics

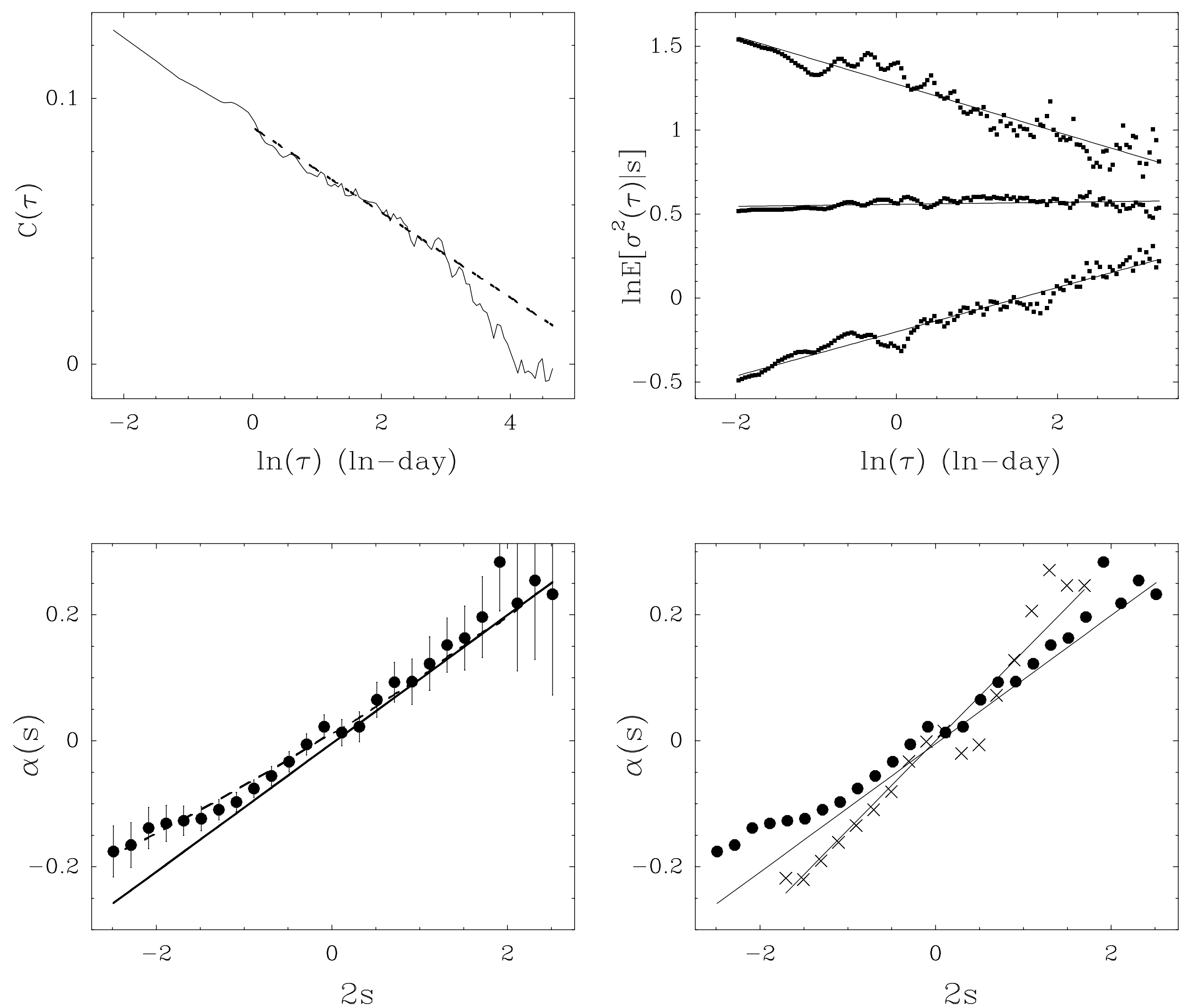
(Sornette, Malavergne, Muzy, 2002)

- Conditional mean: $\mathbb{E}\left[\sigma^2(t+\tau)|\sigma^2(t) = \sigma_0^2 e^s\right]$
- MRW theory (log-normal approx.):

$$\mathbb{E}\left[\tau, s\right] \simeq E_0 \tau^{\alpha(s)}$$

$$\alpha(s,\tau) = C_\Delta s$$

Empirical estimates for S&P 100 index.



Other applications

Portfolio selection problem

• Problem : Optimal portfolio composition in order to minimize the risk an maximize the return.

• Classical portfolio selection (*Markowitz 1959*)
- The variance fully defines the risk (Gaussian world or quadratic utility function)
- The full information about the market is encoded in the mean returns μ_i and in the covariance matrix β_{ij}.
- The solutions can be computed exactly and are located on the efficient frontier $\mu_p = \mu_0 + \sigma_p^2$.
- No time scale (horizon) dependence (because μ and σ^2 are linear functions of the horizon)
- Optimal portfolios are stable by linear superposition.

$\rightarrow$ CAPM : Relates the mean return of some asset to its covariance with the "market portfolio".

• Non gaussian return fluctuations
- Risk dependent optimization
- Non trivial time scale dependence (multi-period optimization)
- Non linearities

Other applications

Portfolio selection for assets with heavy tails

- α-stable models (*Fama 1965, Walter et al. 1995*)

-"Natural" extension of the gaussian framework ($\alpha = 2$).

- Most of Markowitz results can be generalized

- Multivariate multifractal model ?

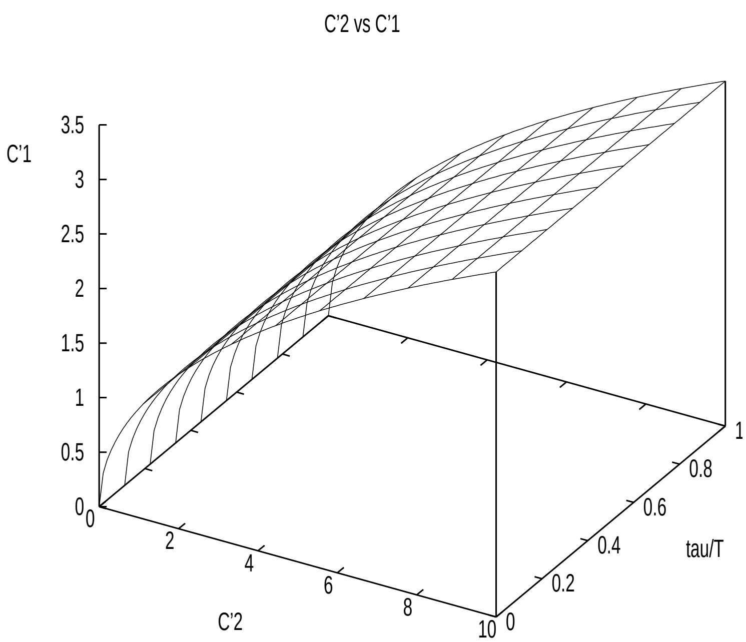
$$\{r_i(k) = \varepsilon_i(k) e^{\Omega_i(k)}\}_{i=1,\dots,N}$$

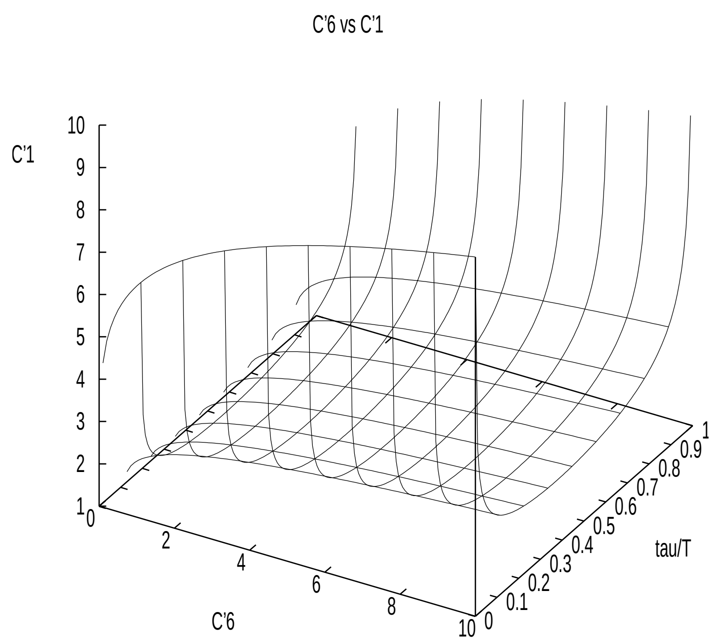
Two "extreme" cases:

- $\Omega_i(k) = \Omega(k)\ \forall i$.

- $\mathbb{C}\mathrm{ov}\,(\Omega_i(k), \Omega_j(k')) = 0$ for $i \neq j$.

-Higher order cumulant utility function





(*Muzy, Sornette, Delour, Arneodo, 2000*)

Other applications
Options and other derivatives

• Derivative security: Financial instrument whose value depends on the value of more basic underlying variables (futures, options,...)

• Option: Contract that give the holder the *right* to buy/sell the underlying asset by a certain date (maturity) for a certain price (strike price)

• Usage: Hedging (reduce risk), speculation (take risk)

• Basic problem of mathematical finance: option pricing and associated hedging strategy.
- Binomial random walk (*Cox & Rubinstein, 1979*)
- Brownian motion (*Black & Scholes, 1971*)

Non Gaussian, heteroskedastic returns $\Rightarrow$ Volatility smile
(Non constant implied volatility as a function of the strike and maturity)

• Problem: Extend Black & Scholes results to more general processes.
-Stochastic volatility (*Hull & White 1988*)
-α-stable markets (*Rachev & al. 1994*)
-Letptokurtic processes, MRW processes
(*Bouchaud & al. 2000, 2002*)

Conclusion and prospects

Multifractal (cascade) models for financial time series:

- Parcimonious models that account for most observed "stylized facts"
- Relatively well known mathematical properties (*Barral, Riedi, Bacry*)
- Stable over "time-aggregation" (scaling)
- Versatility (log-infinitely divisible) (*Barral, Riedi, Bacry*)
- Econometry of multifractal processes (estimation, hypothesis testing,...)
- Financial engineering using multifractals (portfolio, stochastic calculus, options,...)

MRW model (log-normal): Stochastic volatility

- 3 parameters (σ^2, T, λ^2)
- All features can be explained from volatility correlations
- Approx. log-normal "renormalized" volatility ($\lambda^2 \ll 1$)
- Estimation and forecasting
- Multivariate MRW
- Skewed MRW (*Pochard, Bouchaud 2002*)

Conclusion and prospects

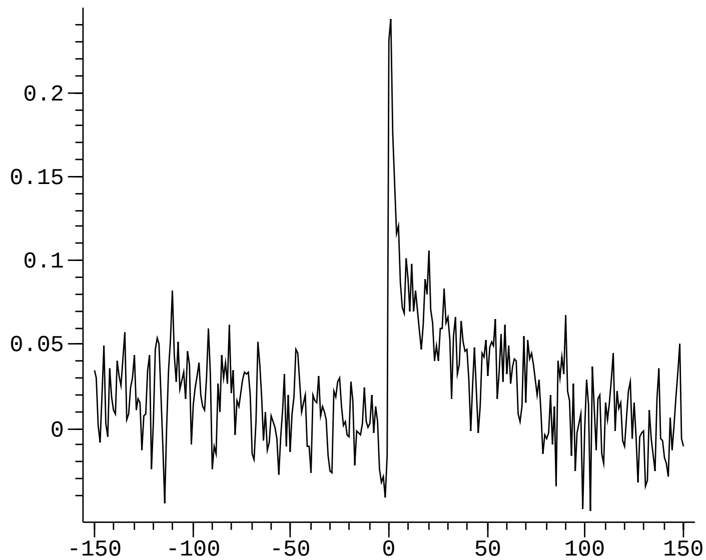
Market microstructure and agent based models

- Order books dynamics (long-ranged)
- Order impact function and volume dynamics
- Agent based models: origin of the cascade (minority games, Lux-Marchesi model,...)

Return / Log-volatility correlations

Bouchaud, Matacz, Potters, 2001, Pochard, Bouchaud 2002

• Correlation $\mathbb{C}\mathrm{ov}\,(-r(t), \omega(t+\tau))$ for the S&P 100 (daily).



• Log-log representation: $\mathbb{C}\mathrm{ov}\,(r(t), \omega(t+\tau)) \simeq -S\tau^{-1/2}$